

# Spectrum of commuting graphs of some classes of finite groups

Jutirekha Dutta and Rajat Kanti Nath\*

Department of Mathematical Sciences,  
Tezpur University, Napaam-784028, Sonitpur, Assam, India.  
Emails: jutirekha.dutta@yahoo.com, rajatkantinath@yahoo.com

**Abstract:** In this paper, we initiate the study of spectrum of the commuting graphs of finite non-abelian groups. We first compute the spectrum of this graph for several classes of finite groups, in particular AC-groups. We show that the commuting graphs of finite non-abelian AC-groups are integral. We also show that the commuting graph of a finite non-abelian group  $G$  is integral if  $G$  is not isomorphic to the symmetric group of degree 4 and the commuting graph of  $G$  is planar. Further it is shown that the commuting graph of  $G$  is integral if the commuting graph of  $G$  is toroidal.

**Key words:** commuting graph, spectrum, integral graph, finite group.

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## 1 Introduction

Let  $G$  be a finite group with centre  $Z(G)$ . The commuting graph of a non-abelian group  $G$ , denoted by  $\Gamma_G$ , is a simple undirected graph whose vertex set is  $G \setminus Z(G)$ , and two vertices  $x$  and  $y$  are adjacent if and only if  $xy = yx$ . Various aspects of commuting graphs of different finite groups can be found in [3, 6, 10, 11, 12, 13]. In this paper, we initiate the study of spectrum of commuting graphs of finite non-abelian groups. Recall that the spectrum of a graph  $\mathcal{G}$  denoted by  $\text{Spec}(\mathcal{G})$  is the set  $\{\lambda_1^{k_1}, \lambda_2^{k_2}, \dots, \lambda_n^{k_n}\}$ , where  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the eigenvalues of the adjacency matrix of  $\mathcal{G}$  with multiplicities  $k_1, k_2, \dots, k_n$  respectively. A graph  $\mathcal{G}$  is called integral if  $\text{Spec}(\mathcal{G})$  contains only integers. It is well known that the complete graph

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\*Corresponding author

$K_n$  on  $n$  vertices is integral. Moreover, if  $\mathcal{G}$  is the disjoint union of some complete graphs then also it is integral. The notion of integral graph was introduced by Harary and Schwenk [9] in the year 1974. A very impressive survey on integral graphs can be found in [5].

We observe that the commuting graph of a non abelian finite AC-group is disjoint union of some complete graphs. Therefore, commuting graphs of such groups are integral. In general it is difficult to classify all finite non-abelian groups whose commuting graphs are integral. As applications of our results together with some other known results, in Section 3, we show that the commuting graph of a finite non-abelian group  $G$  is integral if  $G$  is not isomorphic to  $S_4$ , the symmetric group of degree 4, and the commuting graph of  $G$  is planar. We also show that the commuting graph of a finite non-abelian group  $G$  is integral if the commuting graph of  $G$  is toroidal. Recall that the genus of a graph is the smallest non-negative integer  $n$  such that the graph can be embedded on the surface obtained by attaching  $n$  handles to a sphere. A graph is said to be planar or toroidal if the genus of the graph is zero or one respectively. It is worth mentioning that Afkhami et al. [2] and Das et al. [7] have classified all finite non-abelian groups whose commuting graphs are planar or toroidal recently.

## 2 Computing spectrum

It is well known that the complete graph  $K_n$  on  $n$  vertices is integral and  $\text{Spec}(K_n)$  is given by  $\{(-1)^{n-1}, (n-1)^1\}$ . Further, if  $\mathcal{G} = K_{m_1} \sqcup K_{m_2} \sqcup \cdots \sqcup K_{m_l}$ , where  $K_{m_i}$  are complete graphs on  $m_i$  vertices for  $1 \leq i \leq l$ , then

$$\text{Spec}(\mathcal{G}) = \{(-1)^{\sum_{i=1}^l m_i - l}, (m_1 - 1)^1, (m_2 - 1)^1, \dots, (m_l - 1)^1\}. \quad (2.1)$$

If  $m_1 = m_2 = \cdots = m_l = m$  then we write  $\mathcal{G} = lK_m$  and in that case  $\text{Spec}(\mathcal{G}) = \{(-1)^{l(m-1)}, (m-1)^l\}$ .

In this section, we compute the spectrum of the commuting graphs of different families of finite non-abelian AC-groups. A group  $G$  is called an AC-group if  $C_G(x)$  is abelian for all  $x \in G \setminus Z(G)$ . Various aspects of AC-groups can be found in [1, 7, 14]. The following lemma plays an important role in computing spectrum of commuting graphs of AC-groups.

**Lemma 2.1.** *Let  $G$  be a finite non-abelian AC-group. Then the commuting graph of  $G$  is given by*

$$\Gamma_G = \bigsqcup_{i=1}^n K_{|X_i| - |Z(G)|}$$

where  $X_1, \dots, X_n$  are the distinct centralizers of non-central elements of  $G$ .

*Proof.* Let  $G$  be a finite non-abelian AC-group and  $X_1, \dots, X_n$  be the distinct centralizers of non-central elements of  $G$ . Let  $X_i = C_G(x_i)$  where  $x_i \in G \setminus Z(G)$  and  $1 \leq i \leq n$ . Let  $x, y \in X_i \setminus Z(G)$  for some  $i$  and  $x \neq y$  then, since  $G$  an AC-group, there is an edge between  $x$  and  $y$  in the commuting graph of  $G$ . Suppose that  $x \in (X_i \cap X_j) \setminus Z(G)$  for some  $1 \leq i \neq j \leq n$ . Then  $[x, x_i] = 1$  and  $[x, x_j] = 1$ . Hence, by Lemma 3.6 of [1] we have  $C_G(x) = C_G(x_i) = C_G(x_j)$ , a contradiction. Therefore,  $X_i \cap X_j = Z(G)$  for any  $1 \leq i \neq j \leq n$ . This shows that  $\Gamma_G = \bigsqcup_{i=1}^n K_{|X_i| - |Z(G)|}$ .  $\square$

**Theorem 2.2.** *Let  $G$  be a finite non-abelian AC-group. Then the spectrum of the commuting graph of  $G$  is given by*

$$\{(-1)^{\sum_{i=1}^n |X_i| - n(|Z(G)|+1)}, (|X_1| - |Z(G)| - 1)^1, \dots, (|X_n| - |Z(G)| - 1)^1\}$$

where  $X_1, \dots, X_n$  are the distinct centralizers of non-central elements of  $G$ .

*Proof.* The proof follows from Lemma 2.1 and (2.1).  $\square$

**Corollary 2.3.** *Let  $G$  be a finite non-abelian AC-group and  $A$  be any finite abelian group. Then the spectrum of the commuting graph of  $G \times A$  is given by*

$$\{(-1)^{\sum_{i=1}^n |A|(|X_i| - n|Z(G)|) - n}, (|A|(|X_1| - |Z(G)|) - 1)^1, \dots, (|A|(|X_n| - |Z(G)|) - 1)^1\}$$

where  $X_1, \dots, X_n$  are the distinct centralizers of non-central elements of  $G$ .

*Proof.* It is easy to see that  $Z(G \times A) = Z(G) \times A$  and  $X_1 \times A, X_2 \times A, \dots, X_n \times A$  are the distinct centralizers of non-central elements of  $G \times A$ . Therefore, if  $G$  is an AC-group then  $G \times A$  is also an AC-group. Hence, the result follows from Theorem 2.2.  $\square$

Now we compute the spectrum of the commuting graphs of some particular families of AC-groups. We begin with the well-known family of quasidihedral groups.

**Proposition 2.4.** *The spectrum of the commuting graph of the quasidihedral group  $QD_{2^n} = \langle a, b : a^{2^{n-1}} = b^2 = 1, bab^{-1} = a^{2^{n-2}-1} \rangle$ , where  $n \geq 4$ , is given by*

$$\text{Spec}(\Gamma_{QD_{2^n}}) = \{(-1)^{2^n - 2^{n-2} - 3}, 1^{2^{n-2}}, (2^{n-1} - 3)^1\}.$$

*Proof.* It is well-known that  $Z(QD_{2^n}) = \{1, a^{2^{n-2}}\}$ . Also

$$C_{QD_{2^n}}(a) = C_{QD_{2^n}}(a^i) = \langle a \rangle \text{ for } 1 \leq i \leq 2^{n-1} - 1, i \neq 2^{n-2}$$

and

$$C_{QD_{2^n}}(a^j b) = \{1, a^{2^{n-2}}, a^i b, a^{i+2^{n-2}} b\} \text{ for } 1 \leq j \leq 2^{n-2}$$

are the only centralizers of non-central elements of  $QD_{2^n}$ . Note that these centralizers are abelian subgroups of  $QD_{2^n}$ . Therefore, by Lemma 2.1

$$\Gamma_{QD_{2^n}} = K_{|C_{QD_{2^n}}(a) \setminus Z(QD_{2^n})|} \sqcup \left( \bigsqcup_{j=1}^{2^{n-2}} K_{|C_{QD_{2^n}}(a^j b) \setminus Z(QD_{2^n})|} \right).$$

That is,  $\Gamma_{QD_{2^n}} = K_{2^{n-1}-2} \sqcup 2^{n-2} K_2$ , since  $|C_{QD_{2^n}}(a)| = 2^{n-1}$ ,  $|C_{QD_{2^n}}(a^j b)| = 4$  for  $1 \leq j \leq 2^{n-2}$  and  $|Z(QD_{2^n})| = 2$ . Hence, the result follows from (2.1).  $\square$

**Proposition 2.5.** *The spectrum of the commuting graph of the projective special linear group  $PSL(2, 2^k)$ , where  $k \geq 2$ , is given by*

$$\{(-1)^{2^{3k}-2^{2k}-2^{k+1}-2}, (2^k - 1)^{2^{k-1}(2^k-1)}, (2^k - 2)^{2^k+1}, (2^k - 3)^{2^{k-1}(2^k+1)}\}.$$

*Proof.* We know that  $PSL(2, 2^k)$  is a non-abelian group of order  $2^k(2^k - 1)$  with trivial center. By Proposition 3.21 of [1], the set of centralizers of non-trivial elements of  $PSL(2, 2^k)$  is given by

$$\{xPx^{-1}, xAx^{-1}, xBx^{-1} : x \in PSL(2, 2^k)\}$$

where  $P$  is an elementary abelian 2-subgroup and  $A, B$  are cyclic subgroups of  $PSL(2, 2^k)$  having order  $2^k, 2^k - 1$  and  $2^k + 1$  respectively. Also the number of conjugates of  $P, A$  and  $B$  in  $PSL(2, 2^k)$  are  $2^k + 1, 2^{k-1}(2^k + 1)$  and  $2^{k-1}(2^k - 1)$  respectively. Note that  $PSL(2, 2^k)$  is a AC-group and so, by Lemma 2.1, the commuting graph of  $PSL(2, 2^k)$  is given by

$$(2^k + 1)K_{|xPx^{-1}|-1} \sqcup 2^{k-1}(2^k + 1)K_{|xAx^{-1}|-1} \sqcup 2^{k-1}(2^k - 1)K_{|xBx^{-1}|-1}.$$

That is,  $\Gamma_{PSL(2, 2^k)} = (2^k + 1)K_{2^k-1} \sqcup 2^{k-1}(2^k + 1)K_{2^k-2} \sqcup 2^{k-1}(2^k - 1)K_{2^k}$ . Hence, the result follows from (2.1).  $\square$

**Proposition 2.6.** *The spectrum of the commuting graph of the general linear group  $GL(2, q)$ , where  $q = p^n > 2$  and  $p$  is a prime integer, is given by*

$$\{(-1)^{q^4-q^3-2q^2-q}, (q^2 - 3q + 1)^{\frac{q(q+1)}{2}}, (q^2 - q - 1)^{\frac{q(q-1)}{2}}, (q^2 - 2q)^{q+1}\}.$$

*Proof.* We have  $|GL(2, q)| = (q^2 - 1)(q^2 - q)$  and  $|Z(GL(2, q))| = q - 1$ . By Proposition 3.26 of [1], the set of centralizers of non-central elements of  $GL(2, q)$  is given by

$$\{xDx^{-1}, xIx^{-1}, xPZ(GL(2, q))x^{-1} : x \in GL(2, q)\}$$

where  $D$  is the subgroup of  $GL(2, q)$  consisting of all diagonal matrices,  $I$  is a cyclic subgroup of  $GL(2, q)$  having order  $q^2 - 1$  and  $P$  is the Sylow  $p$ -subgroup of  $GL(2, q)$  consisting of all upper triangular matrices with 1 in the diagonal. The orders of  $D$  and  $PZ(GL(2, q))$  are  $(q - 1)^2$  and  $q(q - 1)$  respectively. Also the number of conjugates of  $D, I$  and  $PZ(GL(2, q))$  in  $GL(2, q)$  are  $\frac{q(q+1)}{2}, \frac{q(q-1)}{2}$  and  $q+1$  respectively. Since  $GL(2, q)$  is an AC-group (see Lemma 3.5 of [1]), by Lemma 2.1 we have  $\Gamma_{GL(2, q)} =$

$$\frac{q(q+1)}{2}K_{|xDx^{-1}|-q+1} \sqcup \frac{q(q-1)}{2}K_{|xIx^{-1}|-q+1} \sqcup (q+1)K_{|xPZ(GL(2, q))x^{-1}|-q+1}.$$

That is,  $\Gamma_{GL(2, q)} = \frac{q(q+1)}{2}K_{q^2-3q+2} \sqcup \frac{q(q-1)}{2}K_{q^2-q} \sqcup (q+1)K_{q^2-2q+1}$ . Hence, the result follows from (2.1).  $\square$

**Theorem 2.7.** *Let  $G$  be a finite group and  $\frac{G}{Z(G)} \cong Sz(2)$ , where  $Sz(2)$  is the Suzuki group presented by  $\langle a, b : a^5 = b^4 = 1, b^{-1}ab = a^2 \rangle$ . Then*

$$\text{Spec}(\Gamma_G) = \{(-1)^{19|Z(G)|-6}, (4|Z(G)| - 1)^1, (3|Z(G)| - 1)^5\}.$$

*Proof.* We have

$$\frac{G}{Z(G)} = \langle aZ(G), bZ(G) : a^5Z(G) = b^4Z(G) = Z(G), b^{-1}abZ(G) = a^2Z(G) \rangle.$$

Observe that

$$\begin{aligned} C_G(a) &= Z(G) \sqcup aZ(G) \sqcup a^2Z(G) \sqcup a^3Z(G) \sqcup a^4Z(G), \\ C_G(ab) &= Z(G) \sqcup abZ(G) \sqcup a^4b^2Z(G) \sqcup a^3b^3Z(G), \\ C_G(a^2b) &= Z(G) \sqcup a^2bZ(G) \sqcup a^3b^2Z(G) \sqcup ab^3Z(G), \\ C_G(a^2b^3) &= Z(G) \sqcup a^2b^3Z(G) \sqcup ab^2Z(G) \sqcup a^4bZ(G), \\ C_G(b) &= Z(G) \sqcup bZ(G) \sqcup b^2Z(G) \sqcup b^3Z(G) \quad \text{and} \\ C_G(a^3b) &= Z(G) \sqcup a^3bZ(G) \sqcup a^2b^2Z(G) \sqcup a^4b^3Z(G) \end{aligned}$$

are the only centralizers of non-central elements of  $G$ . Also note that these centralizers are abelian subgroups of  $G$ . Thus  $G$  is an AC-group. By Lemma 2.1, we have

$$\Gamma_G = K_{4|Z(G)|} \sqcup 5K_{3|Z(G)|}$$

since  $|C_G(a)| = 5|Z(G)|$  and

$$|C_G(ab)| = |C_G(a^2b)| = |C_G(a^2b^3)| = |C_G(b)| = |C_G(a^3b)| = 4|Z(G)|.$$

Therefore, by (2.1), the result follows.  $\square$

**Proposition 2.8.** *Let  $F = GF(2^n)$ ,  $n \geq 2$  and  $\vartheta$  be the Frobenius automorphism of  $F$ , i. e.,  $\vartheta(x) = x^2$  for all  $x \in F$ . Then the spectrum of the commuting graph of the group*

$$A(n, \vartheta) = \left\{ U(a, b) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & \vartheta(a) & 1 \end{bmatrix} : a, b \in F \right\}.$$

*under matrix multiplication given by  $U(a, b)U(a', b') = U(a + a', b + b' + a'\vartheta(a))$  is*

$$\Gamma_{A(n, \vartheta)} = \{(-1)^{(2^n-1)^2}, (2^n - 1)^{2^n-1}\}.$$

*Proof.* Note that  $Z(A(n, \vartheta)) = \{U(0, b) : b \in F\}$  and so  $|Z(A(n, \vartheta))| = 2^n - 1$ . Let  $U(a, b)$  be a non-central element of  $A(n, \vartheta)$ . It can be seen that the centralizer of  $U(a, b)$  in  $A(n, \vartheta)$  is  $Z(A(n, \vartheta)) \sqcup U(a, 0)Z(A(n, \vartheta))$ . Clearly  $A(n, \vartheta)$  is an AC-group and so by Lemma 2.1 we have  $\Gamma_{A(n, \vartheta)} = (2^n - 1)K_{2^n}$ . Hence the result follows by (2.1).  $\square$

**Proposition 2.9.** *Let  $F = GF(p^n)$ ,  $p$  be a prime. Then the spectrum of the commuting graph of the group*

$$A(n, p) = \left\{ V(a, b, c) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} : a, b, c \in F \right\}.$$

*under matrix multiplication  $V(a, b, c)V(a', b', c') = V(a + a', b + b' + ca', c + c')$  is*

$$\Gamma_{A(n, p)} = \{(-1)^{p^{3n}-2p^n-1}, (p^{2n} - p^n - 1)^{p^n+1}\}.$$

*Proof.* We have  $Z(A(n, p)) = \{V(0, b, 0) : b \in F\}$  and so  $|Z(A(n, p))| = p^n$ . The centralizers of non-central elements of  $A(n, p)$  are given by

- (i) If  $b, c \in F$  and  $c \neq 0$  then the centralizer of  $V(0, b, c)$  in  $A(n, p)$  is  $\{V(0, b', c') : b', c' \in F\}$  having order  $|p^{2n}|$ .
- (ii) If  $a, b \in F$  and  $a \neq 0$  then the centralizer of  $V(a, b, 0)$  in  $A(n, p)$  is  $\{V(a', b', 0) : a', b' \in F\}$  having order  $|p^{2n}|$ .
- (iii) If  $a, b, c \in F$  and  $a \neq 0, c \neq 0$  then the centralizer of  $V(a, b, c)$  in  $A(n, p)$  is  $\{V(a', b', ca'a^{-1}) : a', b' \in F\}$  having order  $|p^{2n}|$ .

It can be seen that all the centralizers of non-central elements of  $A(n, p)$  are abelian. Hence  $A(n, p)$  is an AC-group and so

$$\Gamma_{A(n, p)} = K_{p^{2n}-p^n} \sqcup K_{p^{2n}-p^n} \sqcup (p^n - 1)K_{p^{2n}-p^n} = (p^n + 1)K_{p^{2n}-p^n}.$$

Hence the result follows from (2.1).  $\square$

We would like to mention here that the groups considered in Proposition 2.8-2.9 are constructed by Hanaki (see [8]). These groups are also considered in [4], in order to compute their numbers of distinct centralizers.

### 3 Some applications

In this section, we show that the commuting graph of a finite non-abelian group  $G$  is integral if  $G$  is not isomorphic to  $S_4$  and the commuting graph of  $G$  is planar. We also show that the commuting graph of a finite non-abelian group  $G$  is integral if the commuting graph of  $G$  is toroidal. We shall use the following results.

**Theorem 3.1.** *Let  $G$  be a finite group such that  $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ , where  $p$  is a prime integer. Then*

$$\text{Spec}(\Gamma_G) = \{(-1)^{(p^2-1)|Z(G)|-p-1}, ((p-1)|Z(G)|-1)^{p+1}\}.$$

*Proof.* The result follows from Theorem 2.2 noting that  $G$  is an AC-group with  $p+1$  distinct centralizers of non-central elements and all of them have order  $p|Z(G)|$ .  $\square$

**Proposition 3.2.** *Let  $D_{2m} = \langle a, b : a^m = b^2 = 1, bab^{-1} = a^{-1} \rangle$  be the dihedral group of order  $2m$ , where  $m > 2$ . Then*

$$\text{Spec}(\Gamma_{D_{2m}}) = \begin{cases} \{(-1)^{m-2}, 0^m, (m-2)^1\} & \text{if } m \text{ is odd} \\ \{(-1)^{\frac{3m}{2}-3}, 1^{\frac{m}{2}}, (m-3)^1\} & \text{if } m \text{ is even.} \end{cases}$$

*Proof.* Note that  $D_{2m}$  is a non-abelian AC-group. If  $m$  is even then  $|Z(D_{2m})| = 2$  and  $D_{2m}$  has  $\frac{m}{2} + 1$  distinct centralizers of non-central elements. Out of these centralizers one has order  $m$  and the rests have order 4. Therefore  $\Gamma_{D_{2m}} = K_{m-2} \sqcup \frac{m}{2}K_2$ . If  $m$  is odd then  $|Z(D_{2m})| = 1$  and  $D_{2m}$  has  $m+1$  distinct centralizers of non-central elements. In this case, one centralizer has order  $m$  and the rests have order 2. Therefore  $\Gamma_{D_{2m}} = K_{m-1} \sqcup mK_1$ . Hence the result follows from (2.1).  $\square$

**Proposition 3.3.** *The spectrum of the commuting graph of the generalized quaternion group  $Q_{4n} = \langle x, y : y^{2n} = 1, x^2 = y^n, yxy^{-1} = y^{-1} \rangle$ , where  $n \geq 2$ , is given by*

$$\text{Spec}(\Gamma_{Q_{4n}}) = \{(-1)^{3n-3}, 1^n, (2n-3)^1\}.$$

*Proof.* Note that  $Q_{4n}$  is a non-abelian AC-group with  $n+1$  distinct centralizers of non-central elements. Out of these centralizers one has order  $2n$  and the rests have order 4. Also  $|Z(Q_{4n})| = 2$ . Therefore  $\Gamma_{Q_{4n}} = K_{2n-2} \sqcup nK_2$ . Hence the result follows from (2.1).  $\square$

As an application of Theorem 3.1 we have the following lemma.

**Lemma 3.4.** *Let  $G$  be a group isomorphic to any of the following groups*

- (i)  $\mathbb{Z}_2 \times D_8$
- (ii)  $\mathbb{Z}_2 \times Q_8$
- (iii)  $M_{16} = \langle a, b : a^8 = b^2 = 1, bab = a^5 \rangle$
- (iv)  $\mathbb{Z}_4 \rtimes \mathbb{Z}_4 = \langle a, b : a^4 = b^4 = 1, bab^{-1} = a^{-1} \rangle$
- (v)  $D_8 * \mathbb{Z}_4 = \langle a, b, c : a^4 = b^2 = c^2 = 1, ab = ba, ac = ca, bc = a^2cb \rangle$
- (vi)  $SG(16, 3) = \langle a, b : a^4 = b^4 = 1, ab = b^{-1}a^{-1}, ab^{-1} = ba^{-1} \rangle$ .

Then  $\text{Spec}(\Gamma_G) = \{(-1)^9, 3^3\}$ .

*Proof.* If  $G$  is isomorphic to any of the above listed groups, then  $|G| = 16$  and  $|Z(G)| = 4$ . Therefore,  $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ . Thus the result follows from Theorem 3.1.  $\square$

The next lemma is also useful in this section.

**Lemma 3.5.** *Let  $G$  be a non-abelian group of order  $pq$ , where  $p$  and  $q$  are primes with  $p \mid (q-1)$ . Then*

$$\text{Spec}(\Gamma_G) = \{(-1)^{pq-q-1}, (p-2)^q, (q-2)^1\}.$$

*Proof.* It is easy to see that  $|Z(G)| = 1$  and  $G$  is an AC-group. Also the centralizers of non-central elements of  $G$  are precisely the Sylow subgroups of  $G$ . The number of Sylow  $q$ -subgroups and Sylow  $p$ -subgroups of  $G$  are one and  $q$  respectively. Therefore, by Lemma 2.1 we have  $\Gamma_G = K_{q-1} \sqcup qK_{p-1}$ . Hence, the result follows from (2.1).  $\square$

Now we state and proof the main results of this section.

**Theorem 3.6.** *Let  $\Gamma_G$  be the commuting graph of a finite non-abelian group  $G$ . If  $G$  is not isomorphic to  $S_4$  and  $\Gamma_G$  is planar then  $\Gamma_G$  is integral.*

*Proof.* By Theorem 2.2 of [2] we have that  $\Gamma_G$  is planar if and only if  $G$  is isomorphic to either  $D_6, D_8, D_{10}, D_{12}, Q_8, Q_{12}, \mathbb{Z}_2 \times D_8, \mathbb{Z}_2 \times Q_8, M_{16}, \mathbb{Z}_4 \rtimes \mathbb{Z}_4, D_8 * \mathbb{Z}_4, SG(16, 3), A_4, A_5, S_4, SL(2, 3)$  or  $Sz(2) = \langle a, b : a^5 = b^4 = 1, b^{-1}ab = a^3 \rangle$ .

If  $G \cong D_6, D_8, D_{10}$  or  $D_{12}$  then by Proposition 3.2, one may conclude that  $\Gamma_G$  is integral. If  $G \cong Q_8$  or  $Q_{12}$  then by Proposition 3.3,  $\Gamma_G$  becomes integral. If



$G \cong \mathbb{Z}_2 \times D_8, \mathbb{Z}_2 \times Q_8, M_{16}, \mathbb{Z}_4 \rtimes \mathbb{Z}_4, D_8 * \mathbb{Z}_4$  or  $SG(16, 3)$  then by Lemma 3.4,  $\Gamma_G$  becomes integral.

If  $G \cong A_4 = \langle a, b : a^2 = b^3 = (ab)^3 = 1 \rangle$  then the distinct centralizers of non-central elements of  $G$  are  $C_G(a) = \{1, a, bab^2, b^2ab\}$ ,  $C_G(b) = \{1, b, b^2\}$ ,  $C_G(ab) = \{1, ab, b^2a\}$ ,  $C_G(ba) = \{1, ba, ab^2\}$  and  $C_G(aba) = \{1, aba, bab\}$ . Note that these centralizers are abelian subgroups of  $G$ . Therefore,  $\Gamma_G = K_3 \sqcup 4K_2$  and

$$\text{Spec}(\Gamma_G) = \{(-1)^6, 2^1, 1^4\}.$$

Thus  $\Gamma_G$  is integral.

If  $G \cong Sz(2)$  then by Theorem 2.7, we have

$$\Gamma_G = \{(-1)^{13}, (3)^1, (2)^5\}.$$

Hence,  $\Gamma_G$  is integral.

If  $G$  is isomorphic to

$$\begin{aligned} SL(2, 3) = \langle a, b, c : a^3 = b^4 = 1, b^2 = c^2, \\ c^{-1}bc = b^{-1}, a^{-1}ba = b^{-1}c^{-1}, a^{-1}ca = b^{-1} \rangle \end{aligned}$$

then  $Z(G) = \{1, b^2\}$ . It can be seen that

$$\begin{aligned} C_G(b) &= \{1, b, b^2, b^3\} = \langle b \rangle, \\ C_G(c) &= \{1, c, c^2, c^3\} = \langle c \rangle, \\ C_G(bc) &= \{1, b^2, bc, cb\} = \langle bc \rangle, \\ C_G(a^2b^2) &= \{1, b^2, a, a^2, a^2b^2, ab^2\} = \langle a^2b^2 \rangle, \\ C_G(ac) &= \{1, b^2, ac, ca^2, a^2bc, ab^2c\} = \langle ac \rangle, \\ C_G(ca) &= \{1, b^2, ca, a^2c, ba^2, ab\} = \langle ca \rangle \quad \text{and} \\ C_G(a^2b) &= \{1, b^2, a^2b, ba, b^3a, (ba)^2\} = \langle a^2b \rangle \end{aligned}$$

are the only distinct centralizers of non-central elements of  $G$ . Note that these centralizers are abelian subgroups of  $G$ . Therefore,  $\Gamma_G = 3K_2 \sqcup 4K_4$  and

$$\text{Spec}(\Gamma_G) = \{(-1)^{15}, 1^3, 3^4\}.$$

Thus  $\Gamma_G$  is integral.

If  $G \cong A_5$  then by Proposition 2.5, we have

$$\text{Spec}(\Gamma_G) = \{(-1)^{38}, 1^{10}, 2^5, 3^6\}$$

noting that  $PSL(2, 4) \cong A_5$ . Thus  $\Gamma_G$  is integral.

Finally, if  $G \cong S_4$  then it can be seen that the characteristic polynomial of  $\Gamma_G$  is  $(x-1)^7(x+1)^{10}(x^2-5)^2(x^2-3x-2)$  and so

$$\text{Spec}(\Gamma_G) = \left\{ 1^7, (-1)^{10}, (\sqrt{5})^2, (-\sqrt{5})^2, \left(\frac{3+\sqrt{17}}{2}\right)^1, \left(\frac{3-\sqrt{17}}{2}\right)^1 \right\}.$$

Hence,  $\Gamma_G$  is not integral. This completes the proof.  $\square$

In [2, Theorem 2.3], Afkhami et al. have classified all finite non-abelian groups whose commuting graphs are toroidal. Unfortunately, the statement of Theorem 2.3 in [2] is printed incorrectly. We list the correct version of [2, Theorem 2.3] below, since we are going to use it.

**Theorem 3.7.** *Let  $G$  be a finite non-abelian group. Then  $\Gamma_G$  is toroidal if and only if  $\Gamma_G$  is projective if and only if  $G$  is isomorphic to either  $D_{14}, D_{16}, Q_{16}, QD_{16}, D_6 \times \mathbb{Z}_3, A_4 \times \mathbb{Z}_2$  or  $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ .*

**Theorem 3.8.** *Let  $\Gamma_G$  be the commuting graph of a finite non-abelian group  $G$ . Then  $\Gamma_G$  is integral if  $\Gamma_G$  is toroidal.*

*Proof.* By Theorem 3.7 we have that  $\Gamma_G$  is toroidal if and only if  $G$  is isomorphic to either  $D_{14}, D_{16}, Q_{16}, QD_{16}, D_6 \times \mathbb{Z}_3, A_4 \times \mathbb{Z}_2$  or  $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ .

If  $G \cong D_{14}$  or  $D_{16}$  then by Proposition 3.2, one may conclude that  $\Gamma_G$  is integral. If  $G \cong Q_{16}$  then by Proposition 3.3,  $\Gamma_G$  becomes integral. If  $G \cong QD_{16}$  then by Proposition 2.4,  $\Gamma_G$  becomes integral. If  $G \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$  then  $\Gamma_G$  is integral, by Lemma 3.5. If  $G$  is isomorphic to  $D_6 \times \mathbb{Z}_3$  or  $A_4 \times \mathbb{Z}_2$  then  $\Gamma_G$  becomes integral by Corollary 2.3, since  $D_6$  and  $A_4$  are AC-groups. This completes the proof.  $\square$

We shall conclude the paper with the following result.

**Proposition 3.9.** *Let  $\Gamma_G$  be the commuting graph of a finite non-abelian group  $G$ . Then  $\Gamma_G$  is integral if the complement of  $\Gamma_G$  is planar.*

*Proof.* If the complement of  $\Gamma_G$  is planar then by Proposition 2.3 of [1] we have that  $G$  is isomorphic to either  $D_6, D_8$  or  $Q_8$ . If  $G \cong D_6$  or  $D_8$  then by Proposition 3.2,  $\Gamma_G$  is integral. If  $G \cong Q_8$  then by Proposition 3.3,  $\Gamma_G$  becomes integral. This completes the proof.  $\square$

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